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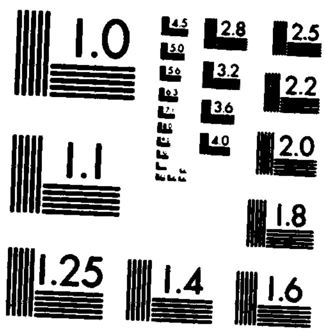

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MRC Technical Summary Report #2516

A GENERALIZATION OF DENJOY'S THEOREM ON DIFFEOMORPHISMS OF THE CIRCLE

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May 1983

(Received April 4, 1983)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

UNIVERSITY OF WISCONSIN-MADISON
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ABSTRACT

We discuss an attempt to generalize Denjoy's theorem on circle maps to maps of two-dimensional manifolds. In particular, we prove that if a C^3 diffeomorphism of a compact two manifold has a point interior to its stable set, then that point must have a periodic ω-limit set, provided the map is 'expansive' off the stable set. We also give a simple example showing that some sort of technical condition is needed, if the assumption 'expansive off the stable set' is to be removed.

AMS (MOS) Subject Classifications: 58F10, 58F22, 58F12

Key Words: Denjoy's theorem, Stable sets

Work Unit Number 1 (Applied Analysis)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and a National Science Foundation Postdoctoral fellowship.

SIGNIFICANCE AND EXPLANATION

Discrete dynamical systems on manifolds, i.e., iterates of a map from a manifold to itself, can serve as simple qualitative models of real systems in physics, biology, and other sciences. Even maps on manifolds of low dimension (one and two) have been used as models since complicated *dynamics* often appears already in this setting. Often, when one is studying a model of a physical or biological system, one is particularly interested in the "stable" orbits, i.e. if a small error is made in setting the initial conditions, ~~we would hope~~ it would be hoped that *this* error *would* not *grow* with time, and, even better, that it would decay with time. Such stable orbits have long been observed, e.g. stable fixed points, periodic orbits form the most elementary examples. It was shown by Denjoy that for maps which are sufficiently smooth diffeomorphisms of the circle, these are the only examples of stable orbits, i.e. any stable orbit must be asymptotic to a periodic orbit.

In this report ~~we~~ the author attempts to generalize this theorem to diffeomorphisms on two-dimensional manifolds. The theorem ~~we give~~ given requires the extra condition that the map be "expansive" off the stable set, and ~~we present~~ presented is an easy example which shows that some additional technical condition will be necessary if "expansive" is to be removed.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A GENERALIZATION OF DENJOY'S THEOREM ON DIFFEOMORPHISMS OF THE CIRCLE

Glen Richard Hall

Introduction. When considering discrete dynamical systems as models of real processes it is important to know which initial conditions are stable under small perturbations, i.e., if a small change is made in the initial state, what will be the effect after many iterates. "Sensitive dependence on initial conditions", where a small change in initial state makes a large change in the asymptotic behavior of the system is commonly observed (see [5]). Alternately, one frequently finds stable orbits which are periodic or fixed points of a system. Finally, it is clear that we could construct continuous maps of metric spaces which have stable orbits, i.e. orbits where a small change in the initial condition will die out under iteration, but such that the orbit is not periodic or asymptotic to a periodic orbit.

Surprisingly, perhaps, the existence of examples of the third type puts, in certain circumstances, limitations on the differentiability of the system. In particular, if S^1 is the circle and $f : S^1 \to S^1$ is a C^2 diffeomorphism of the circle then if an orbit is 'stable' under a slight change in initial conditions (i.e. if the perturbation dies out with time), then the orbit is asymptotic to a periodic orbit. (This theorem of Denjoy [1] is the center piece of a classification for maps of the circle, see [4].) Denjoy also constructed C^1-diffeomorphisms of S^1 which have no periodic orbits, but which have 'stable' orbits as described above. This report is a, not particularly successful attempt to generalize the theorem of Denjoy to two dimensions.

Recently, a difficult and beautiful construction of Harrison (see [2,3]) gives an example which, although constructed for a different purpose, seems to imply that the "correct" generalization of Denjoy's theorem is that in two-dimensions we should replace C^2 by C^3, i.e., that for a C^3-diffeomorphism of a two-dimensional compact manifold,

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every 'stable' orbit is asymptotic to a periodic orbit. We give an easy example which indicates that this is not the case, and we give a theorem which states that if the map is 'expanding' away from the stable orbit then such a theorem does hold. It seems to the author that there is still considerable room between the theorem and the example.

<u>Definitions and Notations</u>. Let M be a metric space with metric ρ. In the sections which follow the space M will be a smooth manifold; however the following definitions make sense for any metric space.

<u>Definition:</u> Suppose $f : M \to M$ is continuous. Then we let

1) $f^0 \equiv$ identity, $f^n = f \circ f^{n-1}$

2) the ω-limit set of a point $x \in M$ is

$$\omega(x) = \bigcap_{n>0} \text{closure } \{f^m(x) : m > n\}.$$

<u>Definition:</u> A set $S \subseteq M$ is said to be periodic under a map $f : M \to M$ if there exists $n > 0$ such that for every $x \in S$ we have $f^n(x) = x$ (i.e. every point of S is periodic with period n).

<u>Remark:</u> We do not require that the least period of every point of S be n.

<u>Definition:</u> The stable set of a point $x \in M$ under a map $f : M \to M$ is

$$W^s(x,f) = \{y : \rho(f^n(y), f^n(x)) \to 0 \text{ as } n \to \infty\} .$$

Statement and Proof of the Theorem. As was pointed out in the introduction, the theorem of Denjoy on homeomorphisms of the circle S' can be interpreted as saying that if $f : S' \to S'$ is a sufficiently smooth diffeomorphism then $x \in$ interior $(W^s(x,f))$ implies $\omega(x)$ is a periodic set. The following theorem is an attempt to generalize this to two dimensions. We let in polar coordinates $A = \{(r,\theta) \in \mathbb{R}^2 : 1 \leq r \leq 2\}$ be the annulus and to avoid the necessity for local coordinates we will state the theorem for this manifold, however it will be clear from the proof that the theorem holds for any smooth compact two dimensional manifold. Let μ and ρ denote the usual measure and metric on A respectively.

Theorem: Suppose $f : A \to A$ is a C^3 diffeomorphism and $x_0 \in A$ satisfies

1) $x_0 \in$ interior $(W^s(x_0,f))$

2) $|Df|(y) > 1$ for all $y \notin \bigcup_{n>0} f^n(W^s(x_0,f))$
(where $|Df|$ denotes the Jacobian of f).

Then $\omega(x_0)$ is a periodic set.

Remark: Clearly condition (2) is very restrictive. By the end of the proof certain generalizations will be clear, however they all require that f be "expansive" on a large set of points. The example of the next section shows that some sort of technical condition is necessary if condition (2) is to be removed.

Proof: Suppose $f : A \to A$ is a C^3 homeomorphism and there exists $x_0 \in A$ such that (1), (2) are satisfied. We may assume

$$(*) \qquad f^i(W^s(x_0,f)) \cap f^j(W^s(x_0,f)) = \emptyset \quad \forall i > j > 0 .$$

For suppose (*) does not hold, i.e., $\exists i > j > 0$ and $z \in A$ with $z \in f^i(W^s(x_0,f)) \cap f^j(W^s(x_0,f))$. Then there exists $x_1, x_2 \in W^s(x_0,f)$ with $f^i(x_1) = z = f^j(x_2)$. Hence

$$\rho(f^n(z), f^{n+i}(x_0)) \to 0, \quad \rho(f^n(z), f^{n+j}(x_0)) \to 0 \quad \text{as} \quad n \to \infty$$

which implies

$$(**) \qquad \rho(f^{n+i-j}(x_0), f^n(x_0)) \to 0 \quad \text{as} \quad n \to \infty .$$

Suppose $w \in \omega(x_0)$. Then there exists $n_k \to \infty$ with

$$w = \lim_{k\to\infty} f^{n_k}(x_0) \;.$$

But then by (**)

$$w = \lim_{k\to\infty} f^{n_k+i-j}(x_0) = f^{i-j}(\lim_{k\to\infty} f^{n_k}(x_0)) = f^{i-j}(w)$$

i.e. the set $\omega(x_0)$ is periodic with period $i - j$. Hence it suffices to assume that (*) holds.

Let $\mu_n = \mu(f^n(W^s(x_0,f)))$ and

$$\varepsilon_n = \max_{y\in f^n(W^s(x_0,f))} \; \min_{z\in\partial f^n(W^s(x_0,f))} \rho(z,y) \;.$$

Then $\sum_{n=0}^{\infty} \mu_n < 1$ by *, and by the definition of ε_n we see that for each n there exists $y_n \in f^n(W^s(x_0,f))$ with $\{z : \rho(y_n,z) < \varepsilon_n\} \subseteq f^n(W^s(x_0,f))$, so

$$\sum_{n=0}^{\infty} \pi\varepsilon_n^2 < \sum_{n=0}^{\infty} \mu_n < 1 \;.$$

(See Figure 1.)

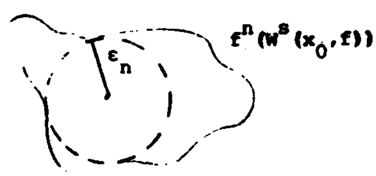

Figure 1

For each $n \geq 0$, let z_n $f^n(W^s(x_0,f))$ be chosen so that

$$|Df|(z_n) = \min_{z\in f^n(W^s(x_0,f))} |Df|(z) \;,$$

so $|Df|(z_n) < \frac{\mu_{n+1}}{\mu_n}$ for all n. Suppose $\frac{\mu_{n+1}}{\mu_n} < 1$. Then z_n is a local min for $|Df|$

and $D|Df|(z_n) = 0$. Fix $n_k \to \infty$ such that $\frac{\mu_{n_k+1}}{\mu_{n_k}} < 1$ for all k and if $\frac{\mu_{n+1}}{\mu_n} < 1$ then $n = n_k$ for some k. Fix $m = n_k$ for some k. Then by the definition of ε_m there exists $w_m \in \partial f^m(W^s(x_0,f))$ with $\rho(z_m,w_m) < \varepsilon_m$ and $|Df|(w_m) = 1$ by (2). But then by the mean value theorem there exists α_m on the segment joining z_m and w_m such that

$$D|Df|(\alpha_m) > \frac{1 - \frac{\mu_{m+1}}{\mu_m}}{\varepsilon_m} .$$

But then, again by the mean value theorem there exists a β_m on the segment joining z_m and α_m such that

$$D^2|Df|(\beta_m) > \frac{1 - \frac{\mu_{m+1}}{\mu_m}}{\varepsilon_m^2} .$$

(See Figure 2.)

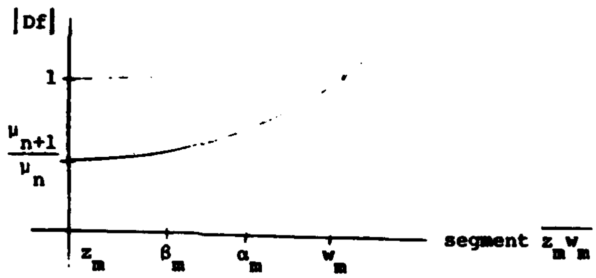

Figure 2

Now since f is C^3 there exists a constant $K > 0$ such that

$$D^2|Df|(\beta_m) < K ,$$

and K is independent of $m = n_k$. Hence we see that for all $k > 0$

$$\frac{1 - \frac{u_{n_k+1}}{u_{n_k}}}{\varepsilon_{n_k}^2} < K$$

i.e.

$$1 - \frac{\mu_{n_k+1}}{\mu_{n_k}} < K\varepsilon_{n_k}^2 .$$

Hence

$$\sum_{k=0}^{\infty} \left(1 - \frac{\mu_{n_k+1}}{\mu_{n_k}}\right) < \sum_{k=0}^{\infty} K\varepsilon_{n_k}^2 < K \sum_{n=0}^{\infty} \varepsilon_n^2 < \infty .$$

But then by a theorem of analysis (Theorem 15.5, Rudin [6])

$$\prod_{k=0}^{\infty} \frac{\mu_{n_k+1}}{\mu_{n_k}} > 0 .$$

But

$$\prod_{k=0}^{N} \frac{\mu_{n_k+1}}{\mu_{n_k}} < \prod_{n=0}^{N} \frac{\mu_{n+1}}{\mu_n} = \frac{\mu_{N+1}}{\mu_0} \to 0 \quad \text{as} \quad N \to \infty$$

and we have a contradiction. Hence equation (*) cannot hold, i.e. $\omega(x_0)$ must be a periodic set. //

<u>Remarks</u>: We could clearly replace condition (2) with

2') $|Df|(y) > 1$ for all $y \in \bigcup_{n \geq 0} \partial f^n(W^s(x_0,f))$,

but this is just as "uncheckable" a condition. Similarly, we could require

2") for each n there exists $y_n \in f^n(W^s(x_0,f))$

such that $\frac{1}{C} < \left(\max_{z \in \partial f^n(W^s(x_0,f))} \rho(z,y_n)\right) / \left(\min_{z \in \partial f^n(W^s(x_0,f))} \rho(z,y_n)\right) < C$

for a constant $C > 0$ independent of n, and for each n there exists $z_n \in \partial f^n W^s(x_0,f)$ with $|Df|(z_n) > 1$.

__An example__. In this section we construct a relatively easy example showing that condition (2), or some similar condition, is necessary in the theorem of the preceding section. In particular we show

__Theorem__: There exists a C^∞ diffeomorphism $f : A \to A$ such that there exists $x_0 \in A$ with $x_0 \in$ interior $W^s(x_0,f)$ but $\omega(x_0)$ is not a periodic set.

__Proof__: Let $f_1 : A \to A$ be defined by

$$f_1(r,\theta) = (r,\hat{f}_1(\theta))$$

where $\hat{f}_1 : S^1 \to S^1$ is a C^∞ diffeomorphism having two fixed points which are nodes at 0 and π, with positive second derivative at each node and $\hat{f}_1''$ near zero. (See Figure 3.)

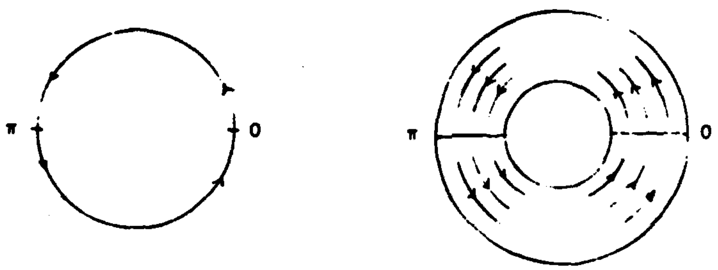

Figure 3

Fix $\theta_1,\theta_2 \in (0,\pi)$, $\theta_3,\theta_4 \in (\pi,2\pi)$ such that

$\hat{f}_1(\theta_1) = \theta_2$, $\hat{f}(\theta_3) = \theta_4$ and let $\eta_1 : (\theta_1,\theta_2) \to [0,1]$, $\eta_2 : (\theta_3,\theta_4) \to [0,1]$ be C^∞ positive bump functions with $\eta_1 = 1$ for $[\bar{\theta}_1,\bar{\theta}_2] \subseteq (\theta_1,\theta_2)$, $\eta_2 = 1$ for $[\bar{\theta}_3,\bar{\theta}_4] \subseteq (\theta_3,\theta_4)$, and $|\eta_2'| < 4$.

Define $g_1 : A \to A$:

$$: (r,\theta) \to (r - \eta_1(\theta) \cdot \frac{1}{2}(r-1),\theta)$$

$$g_2 : A \to A$$

$$: (r,\theta) \to (r,\theta - \eta_2(\theta) \cdot \frac{1}{40} \cdot (\theta - \frac{\theta_4 - \theta_3}{2}))\ .$$

Finally let $h_0 = g_2 \circ g_1 \circ f_1$. (See Figure 4.)

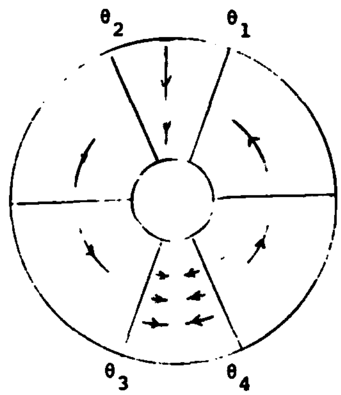

Figure 4

Now, let $B \subseteq A$ be a subset of $(\frac{3}{2},2) \times (\bar{\theta}_1,\bar{\theta}_2)$ (see Figure 5). Then $h_0^n(B)$ limits onto a segment contained in $(\frac{3}{2},1) \times \{\pi\}$. By a C^∞ small perturbation of h_0 near the line $\theta = \pi$, we can create a map h_1 such that h_1 has no fixed points on $(\frac{3}{2},2) \times \{\pi\}$ and there exists $N_1 > 0$ with $h_1^{N_1}(B) \subseteq (\frac{3}{2},2) \times (\bar{\theta}_3,\bar{\theta}_4)$. Now $h_1^n(B)$ will tend, as $n \to \infty$ to the segment $(\frac{3}{2},2) \times \{0\}$. By a C^∞ small perturbation of h_1 near $\theta = 0$ we can create a map h_2 such that there exists $N_2 > 0$ with $h_2^{N_2}(B) \subseteq B$. Continuing in this manner we can create a sequence of C^∞ diffeomorphisms into A converging to the diffeomorphism f from A into A such that $f^n(B)$ is contained in the sets $[1,2] \times (\bar{\theta}_1,\bar{\theta}_2)$ and $[1,2] \times (\bar{\theta}_3,\bar{\theta}_4)$ infinitely often and f is C^∞ close to h_0. Hence each point in the interior of B is in the interior of its stable set. Moreover, for each $x \in B$, $\omega(x)$ includes points in, for example, $\{1\} \times (\bar{\theta}_1,\bar{\theta}_2)$. Hence the ω-limit set is not periodic and the construction is complete.

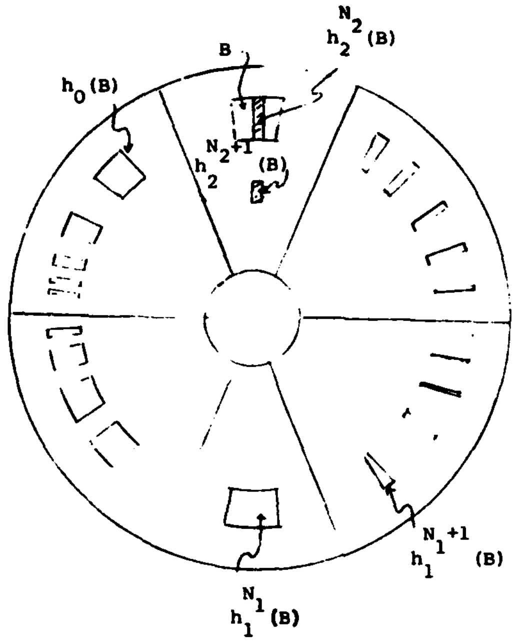

Figure 5

Concluding Remark. It seems that there is still considerable room between the theorem and the example given above. In particular the proof of the theorem is "local" in nature, not using the fact that the images of the stable set return close to one another -- it is this fact which allows the theorem of Denjoy on S' to be improved from "C^2" to "C^1 plus bounded variation of the derivative". One hopes that considerable improvement is possible in the above, perhaps using a more global approach, but we don't know.

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1. REPORT NUMBER 2516 / 2. GOVT ACCESSION NO. A130 575	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A GENERALIZATION OF DENJOY'S THEOREM ON DIFFEOMORPHISMS OF THE CIRCLE	5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Glen Richard Hall	8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below.	12. REPORT DATE May 1983
	13. NUMBER OF PAGES 12
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

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Research Triangle Park
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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Denjoy's theorem
Stable sets

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We discuss an attempt to generalize Denjoy's theorem on circle maps to maps of two-dimensional manifolds. In particular, we prove that if a C^3 diffeomorphism of a compact two manifold has a point interior to its stable set, then that point must have a periodic ω-limit set, provided the map is 'expansive' off the stable set. We also give a simple example showing that some sort of technical condition is needed, if the assumption 'expansive off the stable set' is to be removed.

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